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Evolution of wave packets constructed based on the scattering wave functions for one-dimensional potential

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Abstract – The evolution problem of wave processes in a field of a one-dimensional potential of an arbitrary is considered. We consider the wave packets constructed on a base of linear combinations of the scattering wave functions which are taken with magnitudes of state indices near to resonance tunneling case. It is shown that the wave processes with different parities of asymptotes describe a same wave process for which the group and phase velocities have opposite direction. The conducted consideration allows to state that the bound states can be interpreted as wave packets of the scattering wave functions to be constructed under a certain way.

Keywords: wave process evolution, resonance tunneling

1. Introduction

The problem of description of a wave process evolution for a media with arbitrary changing from point to point linear physical properties always aroused a great interest. Despite an increase of a number of works devoting to this problem the interest to description of a wave process evolution does not decrease, more over it has tendency to increase. So, the practical problem of miniaturization and quickening of work parameters of different electronic and optoelectronic devises strongly connects with the wave problem. It is important to determine the structure and composite features of physical systems where a wave perturbation fast passes into a volume [1-7]. From the theoretical point of view this problem interest is motivated by the necessity of determination of quantity characteristics for a wave evolution process. A wave perturbation is an object of many degrees of freedom. In a uniform media case the propagation of a wave process can be characterized by means of two types of velocities, such as group and phase velocities. However, in the case of an inhomogeneous media a process quantity description is more complicated problem of fundamental character. So, it is well known the Hartman paradox about a tunneling time of a quantum particle passing through a potential barrier [8-16].

Further we will discuss the wave evolution problem on the base of matter waves propagating in a one-dimensional non-regular media with time independent parameters. It is known that for this case any wave process should take place in accordance with the one-dimensional time dependent Schrödinger equation [17];

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=\hat{H}\Psi(x,t),\ \hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x), \tag{1}$$

where $U(x)$ is a particle potential energy, which is suggested to be a function of a one space coordinate only. It is well known that the wave equation (1) has a unique solution if the wave function is a given one for any time moment.

Considering a wave process staring with a moment $t=0$ a time evolution of a wave process can be presented by means of the following formula (see Appendix):

$$\Psi(x,t)=\left[1+\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\hat{H}}{i\hbar}t\right)^m\right]\Psi(x,0)=\exp\left\{-i\frac{\hat{H}}{\hbar}t\right\}\Psi(x,0), \tag{2}$$

where $\Psi(x,0)$ is an initial form of a wave perturbation. In accordance with Eq. (2) the whole character of a wave evolution process is defined by an initial form of a wave excitation.

A wave problem is defined by an initial form of a wave excitation $\Psi(x,0)$, which is given independently on a wave equation, namely, on the media parameters, which are presented in Eq.(1) by means of a potential energy $U(x)$. Although in the wave problem the functions $\Psi(x,0)$ and $U(x)$ are formally independent for a correct statement of a physical problem a certain connection between them would take place. The physical statement of a problem suggests the following question availability: an evolution of what wave process is needed to consider. So, for one potential the given form of $\Psi(x,0)$ can correspond to a wave process dominantly developing in a one direction, for another potential the given form of the initial perturbation can excite a wave process propagating in both directions and so on. In other words from the mathematical point of view the functions $\Psi(x,0)$ and $U(x)$ $U(x)$ are independent, but from the point of view of the problem physical statement these functions are required to consider suggesting a certain connection between them.

Usually the initial form of a wave perturbation is given by mean of a spectral expansion of the function $\Psi(x,0)$ on a base of some set of orthogonal functions. The most important and famous base of a spectral expansion is the Fourier expansion, which is done on the base of both harmonicon time and on a space coordinate functions;

$$\exp\{iqx - iE(q)t/\hbar\}$$

where $E(q) = \hbar^2 q^2 / 2m$. For this case the expansion coefficients will be functions of t;

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \theta(q,t)\exp\{iqx - iE(q)t/\hbar\}dq, \tag{3}$$

where $\theta(k,t)$ gives the expansion coefficients of the Fourier waves for different values of k. Note that the spectral composition of the Fourier waves in dependence on time are changed and the form of the initial perturbation $\Psi(x,0)$ is given by the magnitudes of the expansion coefficients at the initial moment of time: $\theta(k,0)$.The dependence of the expansion coefficients on time is essentially complicates the physical statement of a problem since for many cases it does not allow predict the possible evolution of a wave process. So, one should choice such base of expansion that the expansion coefficients will not be depend of time.

It is clear that it can be in that case only when the expansion base is defined by the eigenfunctions of the Hamiltonian (see Eq. (1)). Considering $\Psi(x,t) = \varphi(x,k)\exp\left\{-i\frac{\hbar k^2}{2m}t\right\}$ for the function $\varphi(x,k)$ one can write down the stationary Schrödinger equation

$$\frac{d^2\varphi(x,k)}{dx^2} + \left[k^2 - u(x)\right]\varphi(x,k) = 0, \tag{4}$$

where $k = \sqrt{2mE}/\hbar$, $u(x) = 2mU(x)/\hbar^2$ and E, $U(x)$ are total and potential energies. So the problem of consideration of a wave process evolution is reduced to solution of Eq.(4). In accordance with the above mentioned for a wave process evolution one can write:

$$\Psi(x,t) = \int_{-\infty}^{+\infty} v(k)\varphi(x,k)\exp\{-iE(k)t/\hbar\}dk. \tag{5}$$

Note that in contrast to Eq. (3)here the expansion coefficients $v(k)$ do not depend on time due to expansion is done over the Hamiltonian eigenfunctions: $\varphi(x,k)$.

2. Some properties of solutions the stationary Schrödinger equation

To use solutions $\varphi(x,k)$ of Eq. (4)for description of a wave process evolution these functions should be normalized and with respect to each other should be orthogonal in Hilbert space;

$$\int_{-\infty}^{+\infty} \varphi^*(x,k)\varphi(x,k')dx = \delta(k-k'),\; v(k) = \int_{-\infty}^{+\infty} \Psi(x,0)\varphi^*(x,k)dx. \tag{6}$$

It is easy to check that in this case expansion can be done only and the equality takes place:

$$\int_{-\infty}^{+\infty} \Psi(x,t)\Psi^*(x,t)dx = \int_{-\infty}^{+\infty} v(k)v^*(k)dk = 1. \quad (7)$$

Here the equality to unity is taken based on a physical mean of a wave functions.

Further we will consider potentials with the lowest value coinciding with the values of a potential when $x \to \pm\infty$ and take it equals to zero;

$$u(x \to \pm\infty) \to 0. \quad (8)$$

It is clear that for potentials of the form of Eq. (8)the energy spectrum is a continuous one and possible motions have an infinite character.

The asymptotic behavior of the solutions of Eq. (4)when Eq. (8) for a potential takes place can be generally written

$$\varphi(x,k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a^* a + d^* d}} \begin{cases} a\exp\{ikx\} + b(k)\exp\{-ikx\}, & x \to -\infty, \\ c(k)\exp\{ikx\} + d\exp\{-ikx\}, & x \to +\infty. \end{cases} \quad (9)$$

Here, as it was shown in paper [18], for any potential form of Eq. (8) the factor

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a^* a + d^* d}} \quad (10)$$

provides the normalization condition of wave functions of an continues spectrum (see Eq. (6)).

Note that when $k > 0$the quantities a, d are being the amplitudes of the waves converging to a potential and b, c are the amplitudes of the diverging waves. In Eq.(9) we suggest that the amplitudes of the converging waves are initially given quantities, therefore the amplitudes of the diverging waves depend on the parameter$k$.

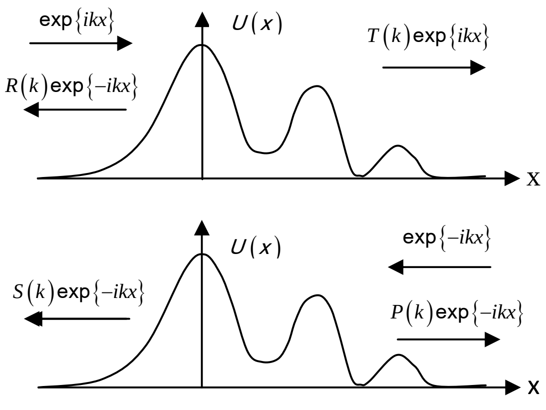

Fig. 1The schematic presentations of the left and right scattering problems.

If one takes in Eq. (9) $a=1$, $d=0$ (a wave falls to potential from its left side and there is no wave falling on a barrier from the right side), then $c=T(k)$, $b=R(k)$ will be the transmission and reflection amplitudes. Then the wave function $\varphi(x,k)$ of the asymptotic behavior Eq.(9) will describe or correspond to the left scattering problem;

$$\varphi_{left}(x,k)=\frac{1}{\sqrt{2\pi}}\begin{cases}\exp\{ikx\}+R(k)\exp\{-ikx\}, & x\to-\infty,\\ T(k)\exp\{ikx\}, & x\to+\infty.\end{cases} \tag{11}$$

For the right scattering problem (a wave falls to a potential from its right side only and there is no wave falling on a barrier from its left side) in Eq. (9)it should be taken $a=0$, $d=1$ and $c=P(k)$, $b=S(k)$, so that

$$\varphi_{right}(x,k)=\frac{1}{\sqrt{2\pi}}\begin{cases}S(k)\exp\{-ikx\}, & x\to-\infty,\\ \exp\{-ikx\}+P(k)\exp\{ikx\}, & x\to+\infty,\end{cases} \tag{12}$$

where $S(k)$, $P(k)$ are the transmission and reflection amplitudes of the right scattering problem.

It is well known that for the transmission and reflection amplitudes of the left and right scattering problems, the following relations take place (see, for example, [19]):

$$T(k)T^{*}(k)+R(k)R^{*}(k)=S(k)S^{*}(k)+P(k)P^{*}(k)=1, \tag{13}$$

$$T(k)=S(k),\ P(k)T^{*}(k)+R^{*}(k)S(k)=0 \tag{14}$$

Note that in Eq. (11), Eq. (12) the factor $1/\sqrt{2\pi}$ is defined by Eq. (10) which provides the normalization condition of the scattering wave functions (see Eq. (6))). It is important to mention that the wave functions of asymptotes Eq. (11), Eq. (12)can be interpreted as scattering wave functions if only the parameter k is considered as positive ($k>0$) .

It should be mentioned that the scattering wave functions are orthogonal with respect to each other as well (see, for example, [18]);

$$\int_{-\infty}^{\infty}\varphi_{left}(x,k)\varphi^{*}_{left}(x,k')dx=\delta(k-k'),\ \int_{-\infty}^{\infty}\varphi_{right}(x,k)\varphi^{*}_{right}(x,k')dx=\delta(k-k') \tag{15}$$

and

$$\int_{-\infty}^{\infty}\varphi_{left}(x,k)\varphi^{*}_{right}(x,k')dx=0\,. \tag{16}$$

This properties of the functions $\varphi_{left}(x,k)$, $\varphi_{right}(x,k)$ are very important to describe different wave processes by means of a linear superposition of them.

3. The wave functions with the mixed asymptotic conditions

Let us consider a wave processes involving the both scattering functions:

$$\phi_+(x,k)=\frac{\varphi_{left}(x,k)+\varphi_{right}(x,k)}{2}\,,\;\phi_-(x,k)=\frac{\varphi_{left}(x,k)-\varphi_{right}(x,k)}{2i}\,. \tag{17}$$

Note, that in accordance with Eq. (17) it can be written

$$\varphi_{left}(x,k)=\phi_+(x,k)+i\phi_-(x,k)\,,\;\varphi_{right}(x,k)=\phi_+(x,k)-i\phi_-(x,k)\,. \tag{18}$$

Using Eq. (15) and Eq. (16) one can check that the both functions $\phi_+(x,k),\ \phi_-(x,k)$ are normalized and orthogonal with respect to each other:

$$\int_{-\infty}^{\infty}\phi_\pm(x,k)\phi_\pm^*(x,k')dx=\delta(k-k'),\;\int_{-\infty}^{\infty}\phi_-(x,k)\phi_+^*(x,k')dx=0. \tag{19}$$

Like to the functions $\varphi_{left}(x,k)$, $\varphi_{right}(x,k)$ any wave process can be presented with help pf the functions $\phi_+(x,k),\ \phi_-(x,k)$;

$$\Psi(x,t)=\int_0^{\infty}\left[v_+(k)\phi_+(x,k)+v_-(k)\phi_-(x,k)\right]\exp\{-iE(k)t/\hbar\}dk\,, \tag{20}$$

where $v_+(k)$, $v_-(k)$ are the coefficients (the functions or the densities) of the expansion spectrum of the function $\Psi(x,t)$ conducted on the basis of the functions $\phi_+(x,k),\ \phi_-(x,k)$ .Using Eq.(19) from Eq. (20)for $v_{left}(k)$, $v_{right}(k)$ one can write

$$v_+(k)=\int_{-\infty}^{+\infty}\Psi(x,0)\phi_+^*(x,k)dx\,,\;v_-(k)=\int_{-\infty}^{+\infty}\Psi(x,0)\phi_-^*(x,k)dx\,. \tag{21}$$

These formulas present the dependences of the spectral compositions of the wave $\phi_+(x,k),\ \phi_-(x,k)$ on initial from of a wave packet perturbation.As we will see below the presentation of the wave packet in the form of Eq. (20) is very useful for description of the resonance tunneling cases and a transmission trough a symmetric potential.

Note if $v_+(k)=v_-(k)$, when the wave packet Eq. (20) includes the left scattering functions only (see Eq. (18));

$$\Psi(x,t)=\int_0^{\infty}2v(k)\varphi_{left}(x,k)\exp\{-iE(k)t/\hbar\}dk\,. \tag{22}$$

In the case when $v_+(k)=v(k)$, $v_-(k)=-v(k)$ the wave packet Eq. (20) will include the right scattering functions only;

$$\Psi(x,t)=\int_0^\infty 2v(k)\varphi_{right}(x,k)\exp\{-iE(k)t/\hbar\}dk\ . \tag{23}$$

It should be mentioned as well that the wave functions $\phi_+(x,k),\ \phi_-(x,k)$ define a full set of functions for description of any wave process propagating in a field of potential form of Eq. (8);

$$\int_0^\infty \left[\phi_+(x,k)\phi_+^*(x',k)+\phi_-(x,k)\phi_-^*(x',k)\right]dk=\delta(x-x')\ . \tag{24}$$

Taking into account Eq. (17) and the asymptotic forms of the functions $\varphi_{left}(x,k)$, $\varphi_{right}(x,k)$ (see Eq. (11), Eq. (12) and the first equality of Eq. (14)) let us write the asymptotic behaviors of the functions $\phi_+(x,k),\ \phi_-(x,k)$;

$$\phi_+(x,k)=\frac{1}{2\sqrt{2\pi}}\begin{cases}\exp\{ikx\}+\left[R(k)+T(k)\right]\exp\{-ikx\}, & x\to-\infty,\\ \exp\{-ikx\}+\left[P(k)+T(k)\right]\exp\{ikx\}, & x\to+\infty\end{cases} \tag{25}$$

and

$$\phi_-(x,k)=\frac{1}{2i\sqrt{2\pi}}\begin{cases}\exp\{ikx\}+\left[R(k)-T(k)\right]\exp\{-ikx\}, & x\to-\infty,\\ -\exp\{-ikx\}-\left[P(k)-T(k)\right]\exp\{ikx\}, & x\to+\infty.\end{cases} \tag{26}$$

As it is seen form Eq. (25) for the both asymptotes of the wave $\phi(x,k)$ there are both reflected and transmitted waves.

For the case of a symmetric potential, when as it is known the equality

$$R(k)=P(k) \tag{27}$$

takes place (see[20]), it is easy to see that $\phi_+(x,k),\ \phi_-(x,k)$ Eq. (25), Eq. (26) are even and odd functions, correspondingly;

$$\phi_+(x,k)=\phi_+(-x,k)=\phi_{odd}(x,k)\ ,\ \phi_-(x,k)=-\phi_-(-x,k)=\phi_{even}(x,k)\ . \tag{28}$$

From the theoretical and practical points of views it is interesting to consider the asymptotic behaviors of the functions $\phi_+(x,k),\ \phi_-(x,k)$ in the case of the so-called resonance tunneling when transmission through a scattering potential takes place with unit probability. So, for the resonance tunneling case the reflection coefficients of the both scattering problems equal to zero: $R(k)R^*(k)=P(k)P^*(k)=0$. Denoting the magnitudes of k corresponding to the cases of the resonance tunneling as k_n one can write:

$$R(k_n)=0\ . \tag{29}$$

This equation is a transcendental one defining the magnitudes of k_n. Since for the magnitudes of $k = k_n$ the module of the transmission coefficient equals to unity $T(k)T^*(k) = 1$, so one can write

$$T(k_n) = \exp\{i\theta(k_n)\}, \tag{30}$$

where $T(k) = \tau(k)\exp\{i\theta(k)\}$, $\tau(k)$ and $\theta(k)$ is are the module and the phase of transmission amplitude: $\tau(k_n) = 1$.

Using Eq. (29) and Eq. (30) for the asymptotic behavior of the functions $\phi_{odd}(x,k)$, $\phi_{even}(x,k)$ Eq. (28) one can get:

$$\phi_{odd}(x,k_n) = \frac{1}{\sqrt{2\pi}}\exp\{i\theta_n/2\}\begin{cases}\cos\{k_n x - \theta_n/2\}, & x \to -\infty,\\ \cos\{k_n x + \theta_n/2\}, & x \to +\infty,\end{cases} \tag{31}$$

$$\phi_{even}(x,k_n) = \frac{1}{\sqrt{2\pi}}\exp\{-i\theta_n/2\}\begin{cases}\sin\{k_n x + \theta_n/2\}, & x \to -\infty,\\ \sin\{k_n x - \theta_n/2\}, & x \to +\infty.\end{cases} \tag{32}$$

where we denoted $\theta(k_n) = \theta_n$. As one can see from Eq. (31), Eq. (32)for an arbitrary potential, when the resonance tunneling case takes place the mixed asymptotic behaviors take even and odd forms.

4. The wave packet contracted on a base of the even and odd scattering functions

Further we will investigate the wave packets with a spectral composition contained the wave functions with magnitudes of k near to the resonance. Let us introduce for the functions $\phi_+(x,k)$, $\phi_-(x,k)$ the same $(v_+(k) = v_-(k) = v(k))$ and a uniform spectral distribution into a small interval around the resonance values k_n. Denoting the distribution $v(k)$ as $v_n(k)$ and taking into account thatthe equality (see Eq. (7))

$$\int_0^{+\infty}\left[v_+(k)v_+^*(k) + v_-(k)v_-^*(k)\right]dk = \int_0^{+\infty} 2\left|v_n(k)\right|^2 dk = 1 \tag{33}$$

should take place one can write:

$$v_n(k) = \frac{1}{2\sqrt{\Delta k}}\begin{cases}0, & k < k_n - \Delta k,\\ 1, & k_n - \Delta k < k < k_n + \Delta k,\\ 0, & k > k_n + \Delta k.\end{cases} \tag{34}$$

In general case the functions $\phi_+(x,k)$, $\phi_-(x,k)$ corresponding to the interval $k_n - \Delta k < k < k_n + \Delta k$ (see Eq. (34)) do not have a certain symmetry. However due to a smallness of

Δk we will suppose that $\phi_+(x,k)$, $\phi_-(x,k)$ by the character of the space dependence are very close to the functions corresponded to the central value of the interval, i.e. $k = k_n$. Note, that the wave functions corresponding to the values of resonance tunneling k_n have a certain symmetry in asymptotes (see Eq. (31), Eq. (32)). So, in accordance with the above mentioned, for the values of k located in the mentioned region the wave functions $\phi_+(x,k)$, $\phi_-(x,k)$ can be considered

$$\phi_+(x,k) \approx \phi_{odd}^n(x,k)\,,\ \phi_-(x,k) \approx \phi_{even}^n(x,k)\,, \tag{35}$$

where

$$\phi_{odd}^n(x,k) = \frac{1}{\sqrt{2\pi}}\exp\{i\theta_n/2\}\begin{cases}\cos\{kx-\theta_n/2\}, & x\to-\infty,\\ \cos\{kx+\theta_n/2\}, & x\to+\infty,\end{cases} \tag{36}$$

$$\phi_{even}^n(x,k) = \frac{1}{\sqrt{2\pi}}\exp\{-i\theta_n/2\}\begin{cases}\sin\{kx+\theta_n/2\}, & x\to-\infty,\\ \sin\{kx-\theta_n/2\}, & x\to+\infty.\end{cases} \tag{37}$$

Here the index n shows these functions are considered when k is near to k_n.

Below we will consider the wave processes constructed on a base of the wave functions of the mixed asymptotesEq. (25), Eq. (26) with values of k near to k_n (see Eq. (36), (37));

$$\Psi(x,t) = \int_0^\infty v_n(k)\phi_n(x,k)\exp\{-iE(k)t/\hbar\}dk. \tag{38}$$

$$\phi_n(x,k) = \phi_{odd}^n(x,k) + \phi_{even}^n(x,k)\,. \tag{39}$$

One can separately consider the wave processes with odd and even asymptotic behaviors;

$$\Psi(x,t) = \Psi_{odd}(x,t) + \Psi_{even}(x,t)\,, \tag{40}$$

where

$$\Psi_{odd}(x,t) = \int_0^\infty v_n(k)\phi_{odd}^n(x,k)\exp\{-iE(k)t/\hbar\}dk\,, \tag{41}$$

$$\Psi_{even}(x,t) = \int_0^\infty v_n(k)\phi_{even}^n(x,k)\exp\{-iE(k)t/\hbar\}dk\,. \tag{42}$$

Forthe functions $E(k)$ we against use a smallness of the k change interval;

$$E(k) \approx \frac{\hbar^2 k_n^2}{2m} + \frac{\hbar^2 k_n}{m}(k-k_n)\,, \tag{43}$$

wherein the second term $\hbar^2 k_n/m = dE(k_n)/dk$ is a magnitude of the derivation of $E(k)$ in the point. Denoting $k - k_n = q$ and taking into account Eq. (43)for the considered wave processes (see Eq. (38)) one can write:

$$\Psi(x,t)=\frac{\exp\{-ik_n u_n^{ph}t\}}{2\sqrt{\Delta k}}\int_{-\Delta k}^{\Delta k}\phi_n(x,k_n+q)\exp\left\{-iqu_n^{gr}t\right\}dq\,, \tag{44}$$

where u_n^{ph}, u_n^{gr} are the phase and group velocities taken when $k=k_n$:

$$u_n^{ph}=\frac{\hbar k_n}{2m},\; u_n^{gr}=\frac{\hbar k_n}{m}\,. \tag{45}$$

Further we will use for the velocities u_n^{ph}, u_n^{gr} the notations

$$u_n^{ph}\equiv u_{ph},\; u_n^{gr}\equiv u_{gr}\,, \tag{46}$$

when these quantities are rewritten without the explicit mentioning of the index n.

Below we investigate how a wave process is changed in time when a space coordinate x tends to $\pm\infty$, i.e. we consider the behavior of $\Psi(x\to\pm\infty,t)$. So, in accordance with Eq.(36) and Eq.(44) for the case of the wave process of the odd asymptote one can write:

$$\Psi_{odd}(x\to-\infty,t)=\frac{\exp\{i\theta_n/2\}}{2\sqrt{2\pi\Delta k}}\exp\{-ik_n u_{ph}t\}\int_{-\Delta k}^{\Delta k}\cos\left((k_n+q)x-\frac{\theta_n}{2}\right)\exp\left\{-iqu_{gr}t\right\}dq\,, \tag{47}$$

$$\Psi_{odd}(x\to+\infty,t)=\frac{\exp\{i\theta_n/2\}}{2\sqrt{2\pi\Delta k}}\exp\{-ik_n u_{ph}t\}\int_{-\Delta k}^{\Delta k}\cos\left((k_n+q)x+\frac{\theta_n}{2}\right)\exp\left\{-iqu_{gr}t\right\}dq\,. \tag{48}$$

Using Eq. (37) and Eq.(44) for the wave process of the even asymptote one get:

$$\Psi_{even}(x\to-\infty,t)=\frac{\exp\{-i\theta_n/2\}}{2\sqrt{2\pi\Delta k}}\exp\{-ik_n u_{ph}t\}\int_{-\Delta k}^{\Delta k}\sin\left((k_n+q)x+\frac{\theta_n}{2}\right)\exp\left\{-iqu_{gr}t\right\}dq\,, \tag{49}$$

$$\Psi_{even}(x\to+\infty,t)=\frac{\exp\{-i\theta_n/2\}}{2\sqrt{2\pi\Delta k}}\exp\{-ik_n u_{ph}t\}\int_{-\Delta k}^{\Delta k}\sin\left((k_n+q)x-\frac{\theta_n}{2}\right)\exp\left\{-iqu_{gr}t\right\}dq\;. \tag{50}$$

The obtained integrals Eq. (47) - Eq. (50) are very easily calculated. So for the odd asymptote process one can get:

$$\Psi_{odd}(x\to-\infty,t)=\frac{1}{2\sqrt{2\pi\Delta k}}\left(\frac{\sin\left(\Delta k(x-u_{gr}t)\right)}{(x-u_{gr}t)}\exp\{-ik_n(x+u_{ph}t)+i\theta_n\}+\right.$$
$$\left.+\frac{\sin\left(\Delta k(x+u_{gr}t)\right)}{(x+u_{gr}t)}\exp\{ik_n(x-u_{ph}t)\}\right), \tag{51}$$

$$\Psi_{odd}(x\to-\infty,t)=\frac{1}{2\sqrt{2\pi\Delta k}}\left(\frac{\sin\left(\Delta k(x-u_{gr}t)\right)}{(x-u_{gr}t)}\exp\{-ik_n(x+u_{ph}t)\}+\right.$$
$$\left.+\frac{\sin\left(\Delta k(x+u_{gr}t)\right)}{(x+u_{gr}t)}\exp\{ik_n(x-u_{ph}t)+i\theta_n\}\right), \tag{52}$$

It is known that:

$$\lim_{\Delta k\to\infty}\frac{\sin\left(\Delta k(x\pm u_{gr}t)\right)}{(x\pm u_{gr}t)}=\pi\delta(x\pm u_{gr}t)\,, \tag{53}$$

and for finite values of Δk the main maximums of these functions correspond to the condition $x\pm u_{gr}t=0$. Denoting $x\pm u_{gr}t=a$ ii is easy to see are essentially differ from zero;

$$\lim_{a\to\pm\infty}\frac{\sin\left(\Delta k(x\pm u_{gr}t)\right)}{(x\pm u_{gr}t)}\to 0\,. \tag{54}$$

In accordance with Eq.(53), Eq. (54) in Eq. (51) the second term will differ from zero (note that in this case x takes negative values and t is positive always). It is clear that for Eq. (52) the first term will differ from x (note that here x takes positive values). On a base of the above mentioned for the wave function of the odd asymptotic one can write:

$$\Psi_{odd}(x,\,t)=\frac{1}{2\sqrt{2\pi\Delta k}}\begin{cases}\dfrac{\sin\left(\Delta k(x+u_{gr}t)\right)}{(x+u_{gr}t)}\exp\{ik_n(x-u_{ph}t)\},\;\; x\to-\infty,\\[2ex] \dfrac{\sin\left(\Delta k(x-u_{gr}t)\right)}{(x-u_{gr}t)}\exp\{-ik_n(x+u_{ph}t)\},\;\; x\to+\infty.\end{cases} \tag{55}$$

It is interesting to note that for the both asymptotes the group and phase velocities have different directions.

For the wave process constructed on a base of the wave functions of the even asymptote from Eq. (49), Eq. (50)

$$\begin{aligned}\Psi_{even}(x\to-\infty,t)=\frac{1}{2\sqrt{2\pi\Delta k}}\Bigg(&i\frac{\sin\left(\Delta k(x-u_{gr}t)\right)}{(x-u_{gr}t)}\exp\{-ik_n(x+u_{ph}t)+i\theta_n\}+\\ &-i\,\frac{\sin\left(\Delta k(x+u_{gr}t)\right)}{(x+u_{gr}t)}\exp\{ik_n(x-u_{ph}t)\}\Bigg),\end{aligned} \tag{56}$$

$$\begin{aligned}\Psi_{even}(x\to-\infty,t)=\frac{1}{2\sqrt{2\pi\Delta k}}\Bigg(&-i\frac{\sin\left(\Delta k(x-u_{gr}t)\right)}{(x-u_{gr}t)}\exp\{-ik_n(x+u_{ph}t)\}+\\ &+\,i\frac{\sin\left(\Delta k(x+u_{gr}t)\right)}{(x+u_{gr}t)}\exp\{ik_n(x-u_{ph}t)+i\theta_n\}\Bigg).\end{aligned} \tag{57}$$

Taking into account Eq. (53), Eq.(54) for the given wave process it takes place:

$$\Psi_{even}(x,\,t)=\frac{-i}{2\sqrt{2\pi\Delta k}}\begin{cases}\dfrac{\sin\left(\Delta k(x+u_{gr}t)\right)}{(x+u_{gr}t)}\exp\{ik_n(x-u_{ph}t)\},\;\;x\to-\infty,\\ \dfrac{\sin\left(\Delta k(x-u_{gr}t)\right)}{(x-u_{gr}t)}\exp\{-ik_n(x+u_{ph}t)\},\;\;x\to+\infty.\end{cases}\tag{58}$$

Comparing Eq. (55) with Eq. (58)it is seen that the functions $\Psi_{odd}(x,t)$, $\Psi_{even}(x,t)$ corresponding to the wave functions with different parity of asymptotes up to a phase factor equal to each other;

$$\Psi_{even}(x,\,t)=i\Psi_{odd}(x,\,t)\,.\tag{59}$$

This result means that the wave processes coincide: $\left|\Psi_{even}(x,\,t)\right|^2=\left|\Psi_{odd}(x,\,t)\right|^2$. If the concept of degeneracy is generally applicable to wave packets, when one can states that in this case there are no two wave processes and one wave process is possible to generateonly. It is also seen that the time of the wave process full out coming from a potential region are defined by it space size and the group velocity.

Conclusion

In the framework of the given work we discuss a wave process evolution presenting a wave packet constructed on a base of linear combinations of the wave functions corresponding to the left and right scattering problems. These wave functions have asymptotes of the mixed form, when on the both sides of a potential there are waves propagating in negative and positive directions. It was shown that for a case of a symmetric potential and the resonance tunneling cases the asymptotes of the given functions have certain parity:

We show that the asymptotic behaviors of the wave packets constructed on a base of the even asymptotic functions and on a base of the odd asymptotic functions coincide. It allows us to suggest that the bound states can be considered as a product of a wave packet evolution.

Appendix I.

The well-known formula (3) is usually obtained by an operation way, however from a methodology point of view it is interesting reproduce this result on a base of the finite-deference methods laying in the origin of solution methods of differential equations. Writing in Eq. (1) $\partial\Psi(x,t)/\partial t=\left[\Psi(x,t+\Delta t)-\Psi(x,t)\right]/\Delta t$ a wave function magnitude at at ime moment $t=n\Delta t$ can be presented as:

$$\Psi(x,n\Delta t)=\left(1+\frac{\hat{H}}{i\hbar}\Delta t\right)^{n}\Psi(x,0)=$$
$$=\left(1+C_1^n\frac{\hat{H}}{i\hbar}\Delta t+C_2^n\left(\frac{\hat{H}}{i\hbar}\Delta t\right)^2+C_3^n\left(\frac{\hat{H}}{i\hbar}\Delta t\right)^3+\dots+C_n^n\left(\frac{\hat{H}}{i\hbar}\Delta t\right)^n\right)\Psi(x,0), \qquad \text{(I.1)}$$

where C_m^n is the well-known binomial coefficients:

$$C_m^n=\frac{n!}{(n-m)!m!}.$$

Note that if $t=n\Delta t$ is a finite quantity, when $n\to\infty$ it is followed that $\Delta t\to 0$. Taking in Eq. (I.1) $\Delta t=t/n$ one can write:

$$\Psi(x,t)=\left(1+\frac{C_1^n}{n}\frac{\hat{H}}{i\hbar}t+\frac{C_2^n}{n^2}\left(\frac{\hat{H}}{i\hbar}t\right)^2+\frac{C_3^n}{n^3}\left(\frac{\hat{H}}{i\hbar}t\right)^3+\dots+\frac{C_n^n}{n^n}\left(\frac{\hat{H}}{i\hbar}t\right)^n\right)\Psi(x,0). \qquad \text{(I.2)}$$

It is easy to check, that

$$\lim_{n\to\infty}\frac{C_m^n}{n^m}=\frac{1}{m!}. \qquad \text{(I.3)}$$

Taking into account (I.3) and considering in Eq. (I.2) $n\to\infty$ and one can get Eq. (3).

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