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A METHOD FOR SOLVING SILVESTER TYPE PARAMETRIC MATRIX EQUATION

A method for solving Silvester type parametric matrix equations is proposed. The method is a generalization of Bartels-Stuard's algorithm for solving Silvester type autonomous matrix equations. The method is based on differential transformations and is called D-analogue of Bartels-Stuard's algorithm. A model example is also presented.

Keywords: matrix equation, nonautonomous matrix, differential transformations, Bartels-Stuard's algorithm, Schure decomposition.

There are many common problems in science and technology which are reduced to the solution of linear autonomous matrix equations. The problems become more complicated when the system is described by nonlinear and/or nonautonomous equations. There are a lot of methods for solving autonomous matrix equations [1-3] but the nonautonomous case is almost not investigated. The efficiency of differential transformations [4] for solving many problems of nonautonomous systems was shown in many investigations by many authors [5,6]. Taking into account the efficiency of using differential transformations for nonautonomous cases, in this paper, differential transformations are applied for solving Sylvester type non-autonomous matrix equations.

Considering the Silvester parametric matrix equation:

$$A_{mxm}(t)\cdot X_{mxn}(t)+X_{mxn}(t)\cdot B_{nxn}(t)=C_{mxn}(t). \tag{1}$$

According to Bartels-Stuard's algorithm [1], the equation (1) is reduced to:

$$U_a(t)\cdot T_a(t)\cdot U_a^T(t)\cdot X(t)+X(t)\cdot U_b(t)\cdot T_b(t)\cdot U_b^T(t)=C(t), \tag{2}$$

where $U_a(t)$ and $U_b(t)$ are unitary matrices [4,5] ($U_a^T(t)=U_a^{-1}(t)$ and $U_b^T(t)=U_b^{-1}(t)$), $T_a(t)$ and $T_b(t)$ are the upper triangular matrices computed as Schure decomposition [1]:

$$T_a(t)=U_a^T(t)\cdot A(t)\cdot U_a(t), \tag{3}$$

$$T_b(t)=U_b^T(t)\cdot B(t)\cdot U_b(t). \tag{4}$$

Taking into account the unitarity of U matrices and multiplying (2) by $U_a^T(t)$ the left side and by $U_b(t)$ the right side (2) becomes:

$$T_a(t)\cdot U_a^T(t)\cdot X(t)U_b(t)+U_a^T(t)\cdot X(t)\cdot U_b(t)\cdot T_b(t)=U_a^T(t)\cdot C(t)U_b(t). \quad (5)$$

Denoted by

$$\tilde{C}(t)=U_a^T(t)\cdot C(t)\cdot U_b(t), \quad (6)$$

$$Y(t)=U_a^T(t)\cdot X(t)\cdot U_b(t), \quad (7)$$

(5) becomes:

$$T_a(t)\cdot Y(t)+Y(t)\cdot T_b(t)=\tilde{C}(t). \quad (8)$$

The solution of equation (1) can be computed as:

$$X(t)=U_a(t)\cdot Y(t)\cdot U_b^T(t). \quad (9)$$

The differential transformations[4-6] will serve as a main mathematical apparatus for solving this parametric matrix equation, so let us introduce the main expression of differential transormations and some of their important properties:

$$X(K)=\frac{H^K}{K!}\cdot\frac{\partial^K x(t)}{\partial t^K}\bigg|_{t=t_\upsilon},\ K=\overline{0,\infty} \quad \risingdotseq \quad x(t)=\aleph(t,t_\nu,H,X(K)), \quad (10)$$

where $X(K)$ is the image (discrete) of the original $X(t)$; H - the known constant (scale coefficient); $\aleph(\cdot)$ – the known function of the variant for reconstruction of the original $X(t)$; t_ν - the center of approximation. The left side of the expression is called forward transformations, and the right side – the reverse transformation or reconstruction. Here are some of the properties of differential transformations, which will be used for the future computations.

According to differential transformation algebra, the matrix equation (8) in image space will be [4-6]:

$$\sum_{l=0}^{K}\left(T_a(l)\cdot Y(K-l)+Y(l)\cdot T_b(K-l)\right)=\tilde{C}(K),\ K=\overline{0,\infty}, \quad (11)$$

where

$$T_a(K)=\sum_{\substack{p=0,q=0,r=0\\p+q+r=K}}^{p=K,q=K,r=K}\left(U_a^T(p)\cdot A(q)\cdot U_a(r)\right),\ K=\overline{0,\infty}, \quad (12)$$

$$T_b(K) = \sum_{\substack{p=0,q=0,r=0 \\ p+q+r=K}}^{p=K,q=K,r=K} \left(U_b^T(p) \cdot B(q) \cdot U_b(r)\right), \; K = \overline{0,\infty}, \tag{13}$$

$$\tilde{C}(K) = \sum_{\substack{p=0,q=0,r=0 \\ p+q+r=K}}^{p=K,q=K,r=K} \left(U_a^T(p) \cdot C(q) \cdot U_b(r)\right), \; K = \overline{0,\infty}, \tag{14}$$

are the matrix discrete or images of (3), (4) and (6) accordingly.

From (11) it follows:

$$\begin{gathered} T_a(0) \cdot Y(K) + Y(K) \cdot T_b(0) = \\ = \tilde{C}(K) - \sum_{l=0}^{K-1} \left(T_a(K-l) \cdot Y(l) + Y(l) \cdot T_b(K-l)\right), \; K = \overline{0,\infty}. \end{gathered} \tag{15}$$

For each iteration ($K = \overline{0,\infty}$) the elements of *Y* are computed as:

$$\begin{gathered} Y_{ij}(K) = \frac{\tilde{C}_{ij}(K) - \left(\sum\limits_{l=i+1}^{m} T_a(K)_{il} \cdot Y(K)_{lj} + \sum\limits_{f=1}^{j-1} Y(K)_{if} \cdot T_b(K)_{fj}\right)}{T_a(K)_{ii} + T_b(K)_{jj}}, \\ i = \overline{m,1}; \; j = \overline{1,n}; \\ T_a(K)_{ii} + T_b(K)_{jj} \neq 0. \end{gathered} \tag{16}$$

The $U_a(K)$ and $U_b(K)$ ($K = \overline{0,\infty}$) discretes can be calculated as the Eigen vectors' matrices of *A(t)* and *B(t)* previously orthogonalized by D-analogue of the Gramm-Schmitdt process [6].

The $X(K)$ discretes can be calculated from the image of (9):

$$X(K) = \sum_{\substack{p=0,q=0,r=0 \\ p+q+r=K}}^{p=K,q=K,r=K} \left(U_a(p) \cdot Y(q) \cdot U_b^T(r)\right), \; K = \overline{0,\infty}. \tag{17}$$

The original $X(t)$ can be reconstructed as the second part of (10).

Example. Now consider the following matrix equation:

$$A(t)\cdot X(t)+X(t)\cdot B(t)=C(t)=$$

$$=\begin{bmatrix} t+1 & 1-t \\ -t & t2 \end{bmatrix}\cdot X(t)+X(t)\cdot\begin{bmatrix} \frac{1}{t} & \frac{1}{t-5} \\ \frac{t}{t^2-t-6} & \frac{t}{t^2-t+3} \end{bmatrix}=$$

$$=\begin{bmatrix} \frac{-t^4+t^3+3t^2+t+12}{-t^2+t+6} & \frac{3t^4-15t^3+13t^2-39t-15}{t^3-6t^2+8t-15} \\ \frac{t^3+t^2-t-6}{t^3-t^2-6t} & \frac{t^6-7t^5+13t^4-16t^3+3t^2+14t+3}{t^3-6t^2+8t-15} \end{bmatrix}.$$

According to (10) the discretes of *A(t)*, *B(t)* and *C(t)* (at $t_\nu=1.5;\ H=1;$) are as follows:

$$A(0)=\begin{bmatrix} 2.5 & -0.5 \\ -1.5 & 2.25 \end{bmatrix};\ A(1)=\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix};\ A(2)=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\ A(K)=\{0\},\ K\ge 3.$$

$$B(0)=\begin{bmatrix} 0{,}66667 & -0{,}28571 \\ -0{,}28571 & 0{,}40000 \end{bmatrix},\ B(1)=\begin{bmatrix} -0{,}44444 & -0{,}08163 \\ -0{,}29932 & 0{,}05333 \end{bmatrix},$$

$$B(2)=\begin{bmatrix} 0{,}29630 & -0{,}02332 \\ -0{,}16845 & -0{,}13511 \end{bmatrix},\ B(3)=\begin{bmatrix} -0{,}19753 & -0{,}00666 \\ -0{,}12118 & 0{,}05784 \end{bmatrix},$$

$$B(4)=\begin{bmatrix} 0{,}13169 & -0{,}00190 \\ -0{,}07825 & 0{,}00518 \end{bmatrix},\ B(5)=\begin{bmatrix} -0{,}08779 & -0{,}00054 \\ -0{,}05289 & -0{,}01819 \end{bmatrix},$$

$$B(6)=\begin{bmatrix} 0{,}05853 & -0{,}00016 \\ -0{,}03505 & 0{,}00832 \end{bmatrix},\ B(7)=\begin{bmatrix} -0{,}03902 & -0{,}00004 \\ -0{,}02343 & -0{,}00041 \end{bmatrix},$$

$$B(8)=\begin{bmatrix} 0{,}02601 & -0{,}00001 \\ -0{,}01560 & -0{,}00244 \end{bmatrix}.$$

$$C(0)=\begin{bmatrix} 3.5357 & 6.0714 \\ 0.2381 & -0.0607 \end{bmatrix},\ C(1)=\begin{bmatrix} 1.9660 & 3.1252 \\ -1.1791 & 3.1484 \end{bmatrix},$$

$$C(2)=\begin{bmatrix} 0.2796 & -0.4011 \\ -0.2557 & 3.3273 \end{bmatrix},\ C(3)=\begin{bmatrix} -0.4714 & -0.0238 \\ -0.5478 & 0.9450 \end{bmatrix},$$

$$C(4)=\begin{bmatrix} -0.3168 & 0.0613 \\ -0.1069 & 0.0637 \end{bmatrix},\ C(5)=\begin{bmatrix} -0.2105 & -0.0430 \\ -0.2454 & -0.0226 \end{bmatrix},$$

$$C(6)=\begin{bmatrix}-0.1405 & 0.0018\\ -0.0469 & -0.0059\end{bmatrix},\ C(7)=\begin{bmatrix}-0.0936 & 0.0091\\ -0.1092 & 0.0089\end{bmatrix},$$

$$C(8)=\begin{bmatrix}-0.0624 & -0.0057\\ -0.0208 & -0.0033\end{bmatrix}.$$

According to D-analogue of orthogonalization of the Gram-Schmitdt process [6], the discretes of $U_a(t)$ and $U_b(t)$ are as follows:

$$U_a(0)=\begin{bmatrix}-0.5547 & 0.8321\\ 0.8321 & 0.5547\end{bmatrix},\ U_a(1)=\begin{bmatrix}0.2560 & 0.1707\\ 0.1707 & -0.2560\end{bmatrix},$$

$$U_a(2)=\begin{bmatrix}-0.0919 & -0.1182\\ -0.1182 & 0.0919\end{bmatrix},\ U_a(3)=\begin{bmatrix}0.0182 & 0.0646\\ 0.0646 & -0.0182\end{bmatrix},$$

$$U_a(4)=\begin{bmatrix}0.0065 & -0.0280\\ -0.0280 & -0.0065\end{bmatrix},\ U_a(5)=\begin{bmatrix}-0.0099 & 0.0083\\ 0.0083 & 0.0099\end{bmatrix},$$

$$U_a(6)=\begin{bmatrix}0.0067 & -0.0002\\ -0.0002 & -0.0067\end{bmatrix},\ U_a(7)=\begin{bmatrix}-0.0031 & -0.0020\\ -0.0020 & 0.0031\end{bmatrix},$$

$$U_a(8)=\begin{bmatrix}0.0009 & 0.0018\\ 0.0018 & -0.0009\end{bmatrix}.$$

$$U_b(0)=\begin{bmatrix}0.5372 & -0.8435\\ 0.8435 & 0.5372\end{bmatrix},\ U_b(1)=\begin{bmatrix}0.2638 & 0.1680\\ -0.1680 & 0.2638\end{bmatrix},$$

$$U_b(2)=\begin{bmatrix}-0.3869 & -0.1884\\ 0.1884 & -0.3869\end{bmatrix},\ U_b(3)=\begin{bmatrix}0.0470 & -0.1286\\ 0.1286 & 0.0470\end{bmatrix},$$

$$U_b(4)=\begin{bmatrix}0.2653 & 0.2678\\ -0.2678 & 0.2653\end{bmatrix},\ U_b(5)=\begin{bmatrix}-0.3610 & -0.0864\\ 0.0864 & -0.3610\end{bmatrix},$$

$$U_b(6)=\begin{bmatrix}0.0009 & 0.0018\\ 0.0018 & -0.0009\end{bmatrix},\ U_b(7)=\begin{bmatrix}0.4630 & 0.4036\\ -0.4036 & 0.4630\end{bmatrix},$$

$$U_b(8)=\begin{bmatrix}-0.5449 & 0.0093\\ -0.0093 & -0.5449\end{bmatrix}.$$

For $T_a(t)$ and $T_b(t)$ from (12) and (13), the matrix discretes are:

$$T_a(0)=\begin{bmatrix}3.25 & -1\\ 0 & 1.5\end{bmatrix},\ T_a(1)=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix},\ T_a(2)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ A(2)=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix};$$

$$T_a(K) = \{0\},\ K \geq 3.$$

$$T_b(0) = \begin{bmatrix} 0.2180 & 0 \\ 0 & 0.8486 \end{bmatrix},\ T_b(1) = \begin{bmatrix} -0.2629 & 0.2177 \\ 0 & -0.1282 \end{bmatrix},$$

$$T_b(2) = \begin{bmatrix} -0.2273 & 0.1451 \\ 0 & 0.3885 \end{bmatrix},\ T_b(3) = \begin{bmatrix} 0.1294 & 0.1145 \\ 0 & -0.2691 \end{bmatrix},$$

$$T_b(4) = \begin{bmatrix} -0.0865 & 0.0763 \\ 0 & 0.2234 \end{bmatrix},\ T_b(5) = \begin{bmatrix} -0.1040 & 0.0523 \\ 0 & -0.0020 \end{bmatrix},$$

$$T_b(6) = \begin{bmatrix} 0.1236 & 0.0349 \\ 0 & -0.0568 \end{bmatrix},\ T_b(7) = \begin{bmatrix} -0.0277 & 0.0234 \\ 0 & -0.0109 \end{bmatrix},$$

$$T_b(8) = \begin{bmatrix} -0.1402 & 0.0156 \\ 0 & 0.1638 \end{bmatrix}.$$

The $\tilde{C}(K)$ discretes are calculated from (14):

$$\tilde{C}(0) = \begin{bmatrix} -3.8304 & -0.3491 \\ 5.8838 & 0.1028 \end{bmatrix},\ \tilde{C}(1) = \begin{bmatrix} 1.5541 & 1.0568 \\ 5.3401 & 3.4547 \end{bmatrix},$$

$$\tilde{C}(2) = \begin{bmatrix} 2.2941 & 4.4893 \\ -0.1348 & -0.8432 \end{bmatrix},\ \tilde{C}(3) = \begin{bmatrix} 0.1574 & 0.7549 \\ 0.7562 & -1.8087 \end{bmatrix},$$

$$\tilde{C}(4) = \begin{bmatrix} 1.4498 & -2.1484 \\ 0.0357 & 2.1249 \end{bmatrix},\ \tilde{C}(5) = \begin{bmatrix} -0.0573 & 1.7826 \\ -1.7510 & 0.1310 \end{bmatrix},$$

$$\tilde{C}(6) = \begin{bmatrix} -1.4382 & 0.5874 \\ 1.3535 & -2.5293 \end{bmatrix},\ \tilde{C}(7) = \begin{bmatrix} 1.5134 & -3.0276 \\ 0.7543 & 3.0819 \end{bmatrix},$$

$$\tilde{C}(8) = \begin{bmatrix} 0.5821 & 2.5169 \\ -2.9709 & 0.4562 \end{bmatrix}.$$

The $Y(K)$ discrete is collected from (16):

$$Y(0) = \begin{bmatrix} -0.1170 & -0.0745 \\ 3.4247 & 0.0438 \end{bmatrix},\ Y(1) = \begin{bmatrix} 1.0130 & 0.5937 \\ 1.6389 & 1.1373 \end{bmatrix},$$

$$Y(2) = \begin{bmatrix} -0.2067 & 0.3738 \\ -0.3285 & -1.1520 \end{bmatrix},\ Y(3) = \begin{bmatrix} 0.1429 & -0.4917 \\ 0.5400 & -0.7634 \end{bmatrix},$$

$$Y(4) = \begin{bmatrix} 0.2514 & 0.0236 \\ -0.2054 & 1.2838 \end{bmatrix},\ Y(5) = \begin{bmatrix} -0.3745 & 0.3855 \\ -0.5451 & -0.6625 \end{bmatrix},$$

$$Y(6)=\begin{bmatrix}0.0767 & -0.4476\\ 0.7900 & -1.0591\end{bmatrix},\ Y(7)=\begin{bmatrix}0.4068 & -0.0459\\ -0.0120 & 1.9631\end{bmatrix},$$

$$Y(8)=\begin{bmatrix}-0.4888 & 0.6503\\ -1.2267 & -0.6442\end{bmatrix}.$$

The $X(K)$ discretes are calculated from (17):

$$X(0)=\begin{bmatrix}1.5 & 2.5\\ 1 & 1.5\end{bmatrix},\ X(1)=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\ X(K)=\{0\},\ K\geq 2.$$

The original $X(t)$ is reconstructed using diferential Taylor's transformation [4-6]:

$$X(t)=\begin{bmatrix}t & t+1\\ 1 & t\end{bmatrix}.$$

Conclusion. The solution of the model example by using the developed D-analogue of Bartels-Stuard's method analytically, satisfies the considered Silvester type nonautonomous matrix equation.

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ՍԻԼՎԵՍՏՐԻ ՏԻՊԻ ՊԱՐԱՄԵՏՐԱԿԱՆ ՄԱՏՐԻՑԱՅԻՆ ՀԱՎԱՍԱՐՈՒՄՆԵՐԻ ԼՈՒԾՄԱՆ ԵՂԱՆԱԿ

Առաջարկված է Սիլվեստրի տիպի պարամետրական մատրիցային հավասարումների լուծման եղանակ՝ հիմնված Պուխովի դիֆերենցիալ ձևափոխությունների վրա, որը Սիլվեստրի տիպի ավտոնոմ մատրիցային հավասարումների լուծման Բարտելս-Ստյուարտի եղանակի ընդհանրացումն է և կոչվում է Բարտելս-Ստյուարտի եղանակի Դ-նմանակ: Ներկայացված է մոդելային օրինակ:

Առանցքային բառեր. մատրիցային հավասարում, ոչ-ավտոնոմ մատրից, դիֆերենցիալ ձևափոխություններ, Բարտելս-Ստյուարտի եղանակ, Շուրի դեկոմպոզիցիա:

А.Г. АВЕТИСЯН, В.Р. АВИНЯН, Д.А. КАЗАРЯН

МЕТОД РЕШЕНИЯ ПАРАМЕТРИЧЕСКИХ МАТРИЧНЫХ УРАВНЕНИЙ ТИПА СИЛЬВЕСТРА

Предложен метод решения параметрических матричных уравнений типа Сильвестра, который является обобщением метода Бартельса-Стюарта для решения автономных матричных уравнений типа Сильвестра. Метод основан на дифференциальных преобразованиях Пухова и назван Д-аналогом метода Бартельса-Стюарта. Представлен модельный пример.

Ключевые слова: матричное уравнение, неавтономная матрица, дифференциальные преобразования, метод Бартельса-Стюарта, декомпозиция Шура.